Comment on "Transition to turbulence in a shear flow"

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In this Comment two main issues are addressed that are pertinent to the interpretation of the transition to turbulence for shear type flows on the basis of transient chaos: the fractal character associated with this transition and the precision needed for a reliable numerical modeling. The analysis is based on a simplified model of 19 degrees of freedom for Couette flow, proposed some years ago by Eckhardt and Mersmann. We conclude that the landscape of lifetimes cannot be fractal, and that a very high numerical accuracy is needed to obtain sound results.

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I. INTRODUCTION

In 1997, Schmiegel and Eckhardt [2] studied the dynamics of localized perturbations in plane Couette flow. The numerical model they used, designed to allow accurate long integration times, was based on a pseudospectral method with an expansion of the velocity field in Fourier modes and Legendre polynomials. Among other topics, the authors analyzed the dependence of lifetimes of turbulent orbits on the Reynolds number and the initial data. In particular, they showed how evolutions identified as turbulent depend in a sensitive way on the initial perturbation amplitude. On the basis of their Figs. 3 and 4, they concluded that the landscape of lifetimes exhibits fractal features, certainly a stronger statement than the previous understanding. Calculations were made with 962 dynamically active degrees of freedom (DOF). Two years later, Eckhardt and Mersmann [1] presented a 19-dimensional Galerkin approximation to a parallel shear flow. Section III of that paper was devoted to the dynamics of perturbations, and the results obtained there reinforce, within a simpler frame and thus with more detail, the conclusions of the earlier work (see their Figs. 5–7). Even when the situations discussed by these authors, and by us in this Comment, are mostly related to transient turbulence (transient chaos [3,4]) but not to plain turbulence (chaos), the extreme sensitivity to perturbations is clearly expressed. The associated numerical difficulties require new calculation approaches usually known as *chaos computability* (see, among others, [5,6]: see comments concerning the relations between strange attractors and chaos computability).

This Comment is based on the above mentioned 19-DOF model [1]. We conclude that the landscape of lifetimes cannot be fractal, and that very high numerical accuracy is needed in order to calculate correctly the transient turbulent trajectories as well as their lifetimes. As we will show, simple arguments based on the continuous dependence on initial values of the original problem support completely the first statement. Nevertheless, numerical evidence will be also presented.

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II. DISCUSSION

The calculations presented here have been made with software based on the Adams-Bashforth (AB) algorithm, on the basis of which numerical solutions of ordinary differential equations (ODEs) can be obtained with high precision. Certainly, higher accuracies notably increase computational costs. In the present work the possibilities of a small machine (Pentium III 733 MHz with 640 megabytes of RAM) have been fully exploited. Relative and absolute precision goals up to 21 digits together with a working precision of 33 digits have been used. Under these extreme conditions, the calculations of an orbit for $0 \leqslant t \leqslant 4000$ took about 1 h of CPU time.

All calculations were made with the original variables y, i.e., no calculation in perturbation variables is presented. The low dimensional model discussed in Ref. [1] contains the equilibrium point $(0, 4/\pi^2, 0, 4/9\pi^2, 0, \ldots, 0)$, which is independent of the Reynolds number defined as $\mathrm{Re} = u_0 d/2\nu$. Here $d = \pi$ is the width of the gap between plates, ν the kinematic viscosity, and $u_0 = 1$ the wall velocity of the linear profile with vanishing mean value. This equilibrium point corresponds to an approximation to Couette's flow (see [1]). The value $\mathrm{Re} = 400$ has been adopted everywhere.

Figure 1 shows how the numerical solutions (in particular their by-products, such as lifetimes) depend on precision parameters. Figure 1 (a) depicts calculations with insufficient precision and Fig. 1 (b) a range of precisions for which the results appear to be "acceptable." It should be pointed out that many other similar calculations have been made (not shown here), showing even more pessimistic results. The energy E is evaluated with the formula (see [1])

$$E = \sum_{i=1}^{7} y_i^2 + 2\sum_{i=8}^{19} y_i^2. \tag{1}$$

If, say, $t > 3000$ the energy takes practically the value 0.166 284…, which is that of the equilibrium point. The (unique) initial condition used here was a selected random vector of 19 components, with L_∞ norm less than 1.

The descriptions above deal with "uncontrolled perturbations" originating in numerical errors. Now, adopting previous information about acceptable levels of precision to obtain reliable solutions, the effects of small "controlled

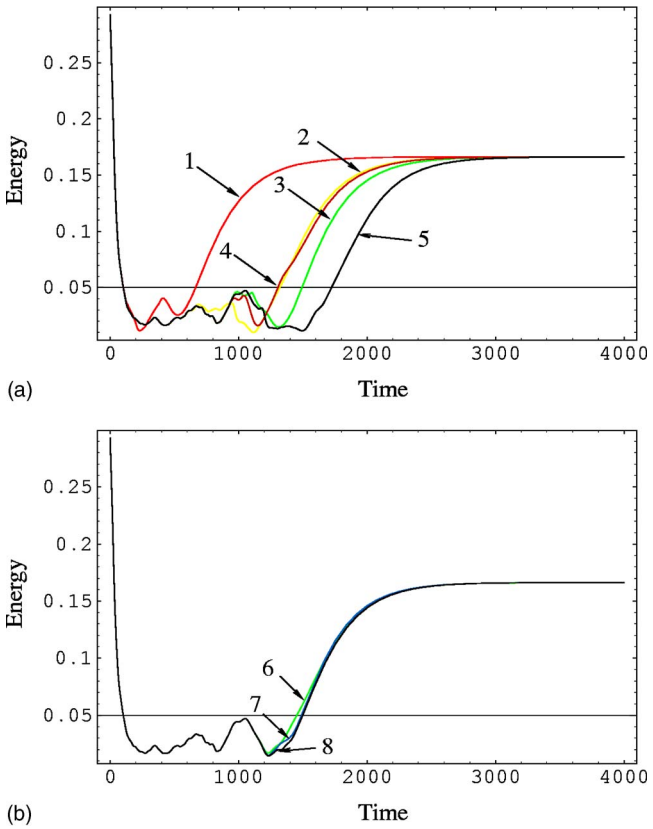

FIG. 1. (Color online) Dependence of lifetimes on precision parameters. Energy (E) vs time (t). Re=400. Numbers between bars mean absolute precision/relative precision/working precision (in decimal digits). (a) Insufficient precision for calculations. 1 (red) 06/06/17; 2 (yellow) 12/12/22; 3 (green) 14/14/24; 4 (brown) 16/16/30; 5 (black) 18/18/30. (b) Acceptable precision. 6 (green) 19/20/30; 7 (blue) 20/20/32; 8 (black) 21/21/33.

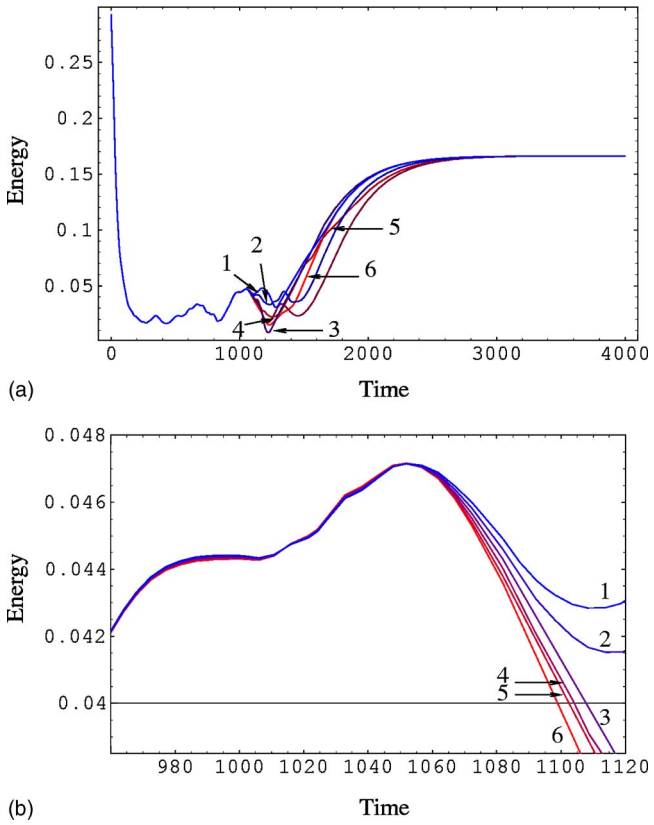

FIG. 2. (Color online) Continuous dependence on initial values. Energy (E) vs time (t). Re=400. Convergence for initial conditions with amplitudes descending from $1/10+10^{-16}$ to $1/10$, in five steps of value -2×10^{-17} (1 to 6 blue to red). The precision used was 20/20/33. (a) Global view. (b) Zoom of the window $(t_1,t_2)\times(E_1,E_2)=(960,1120)\times(0.0385,0.048)$, showing monotonic convergence.

perturbations" on the initial conditions are empirically investigated. To that purpose, the same random initial condition vector as in Fig. 1 was used, to be later modulated slightly by variable amplitude factors (controlled perturbations). Thus, initial conditions change along a small segment of a ray in the phase space, defined by the previous vector. Figure 2 (a) depicts six solutions originated in equidistant initial conditions of amplitudes $1/10+10^{-16}$ to $1/10$, distributed along the given ray. Figure 2 (b) shows a detail in a small window where curves begin to separate. Three features are worth noting. (1) Nearby orbits move close together, until a region where they drastically separate (*diverting solutions*; see [5]). (2) The convergence is monotonic for solutions with t less than 1120, say, for a very small range of amplitudes of value 10^{-16}. (3) In Figs. 1 and 2 (as well as in many other figures corresponding to this work, but not shown here), whenever (controlled or uncontrolled) perturbations diminish, the point where the perturbed curve violently separates from the reference curve moves to the right (see Fig. 4b of Ref. [6] and related text). In other words, the smaller the perturbation, the longer the separation instant. This last observation is true in practically all cases, with some minor exceptions [see, for example, curves in Fig. 1 (upper)]. It also illustrates how the continuous dependence on initial values typically manifests itself in situations close to chaotic dynamics (positive and negative local Lyapunov exponents, i.e., perturbation-sensitive neighborhoods [5]).

The discussion in the previous paragraph leads to the following conclusion. For a given appropriate precision, there exists a sufficiently small interval of amplitudes $[0.1,0.1+\epsilon]$, $0<\epsilon\ll10^{-16}$, containing initial conditions, such that all separation points lie to the right of, say, $t=4000$. Thus, the family of curves like those in Fig. 2 (upper) created in these initial conditions will exhibit a monotonic convergence to the reference curve, which was obtained with amplitude 1/10. The next conclusion reads: The fractal character apparently expressed by successive magnification in Fig. 7 of Ref. [1] (Re=200) terminates for amplitude intervals roughly of order ϵ, with a clear simplification of small scales: a smooth monotonic variation of lifetime with amplitude is expected. The smallest scale discernible in Fig. 7 of [1] is of the order 10^{-9}, to be contrasted with $\epsilon\ll10^{-16}$. This last conclusion should also be qualitatively true for Fig. 4 of Ref. [2] (Re =380). Unfortunately, calculation of such high precision orbits, corresponding to $\epsilon\ll10^{-16}$, was out of our reach. To skip some difficulties, our calculations have been made with Re=400, somewhat higher than those used in the two previous references. However, this value of Re is well within the

transition zone shown in Fig. 5 of Ref. [1]. (Other values of Re have also been used for numerical experiments, leading to the same qualitative conclusions.) Finally, it becomes clear that the transition zone is a region characterized by drastic but smooth variations [7].

Numerical analysis is not necessary to reach such a conclusion. As mentioned in the Introduction, nonfractality is a direct consequence of the continuous dependence on the initial values of the original problem. This is an essential condition with physical relevance. At least for the present case, problems satisfy such a condition of well-posedness. This can be seen by straightforward informal arguments. The vector field of the 19-dimensional model under study (equations not shown here) is polynomial of degree 2, thus locally Lipschitz. All solutions of interest are bounded. It is thus always possible to accommodate appropriate compact sets in the phase space containing them. In this way, the local Lipschitz condition becomes a global one within such sets. Classical theorems for ODEs state the continuous dependence on initial values for any compact set in the extended phase space (that enriched with the time axis), containing the initial point (see, for example, Ref. [8] Chap. 6, Theorem 2 and Corollary, among other texts for ODEs). Thus, if a series of initial conditions tends to a previously given one, as illustrated in Fig. 2, the same is true for the associated lifetimes (as well as for any other quantity of interest, continuously dependent on solutions).

To assess the results here presented, most of the calculations have been reproduced using the Runge-Kutta-Fehlberg algorithm of combined fourth and fifth orders (RKF4-5). This time a Pentium IV, 2.8 GHz with 1 gigabyte of RAM was used. We have observed that if the precisions of both ODE solvers are to be reasonably comparable, about two additional orders of accuracy were needed for RKF4-5. In this way, results and conclusions obtained here with the AB algorithm were essentially confirmed by the RKF4-5. "Essentially" refers to the difficulties and limitations in high precision numerical processing, always present (for example, the extreme sensibility occasionally observed, manifested by variations of the results when solving the same problem in different computers, within the same general conditions).

All the previous discussion has been restricted to the simplified model proposed in Ref. [1], with 19 DOF. It is then reasonable to expect that for the more complex and realistic models used in current practice, with much higher numbers of DOF, the effects of numerical errors will go from bad to worse.

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